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LETTER TO THE EDITOR

Rigorous solution for electromagnetic waves propagating through pre-Cantor sets

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Abstract

We study one-dimensional systems constructed from a segment by employing the Cantor-set rule up to an arbitrary stage of self-similar patterns. The rigorous expression of the transfer matrix to describe the electromagnetic waves propagating through them is presented. As displayed by numerical demonstration, the transmission spectra change drastically with the increase of the stage. At rather high stages the periodicity hidden in the self-similarity comes out. This is the first theoretical description of sharp attenuation in the transmission spectrum of electromagnetic waves propagating through a fractal medium.

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The propagation of electromagnetic waves through a fractal medium with self-similar symmetry has attracted much attention [1, 2]. In particular, light localization in the fractals is interesting and important in the fields of science and technology [3–6]. Quite recently we have discovered the localization of microwaves in the Menger sponge, which is typically a classic fractal in three dimensions [7, 8]. Sharp single dips both in the transmission and reflection spectra were observed at the same specific frequency. In order to understand inherently the mechanism, theoretical approaches are desirable in addition to the numerical ones [9]. As far as we know, no successful analytic study to elucidate the sharp attenuation in the transmission spectrum exists.

On the other hand, from a theoretical point of view one-dimensional systems are attractive because one can expect to obtain rigorous solutions. Indeed, there have been many studies for electromagnetic waves and for Schrödinger waves in one-dimensional systems. The latter go back to the late 1970s and early 1980s in the context of the Anderson localization in the tight binding model [10]. However, for the case of electromagnetic waves there exist two degrees of freedom, amplitude and phase, which make problems to be solved more difficult. Even for the simplest case, the triadic Cantor set, studies have been restricted to numerical [11–13] or approximate [14] attempts. The other studies are limited to those with quasi-periodicity [15, 16]. Therefore we should start with the Cantor set in the first step.

The triadic Cantor set is constructed from an initiator, which is a segment $[a, b]$ with $L = b - a$, by removing its middle third. The procedure of removing the middle third from the remainders is repeated until a required stage n. We call this the nth-stage pre-Cantor set in this letter. The true Cantor set in a mathematical sense appears in the limit of $n \to \infty$.

Let us first prepare the basic formula of the transfer matrix. The electromagnetic waves in a one-dimensional uniform system are generally represented by

$$E(x, t) = \frac{1}{2\pi} \int d\omega\, e^{-i\omega t} \{A_r(\omega)\, e^{ikx} + A_l(\omega)\, e^{-ikx}\}, \tag{1}$$

where $A_r(\omega)(A_l(\omega))$ denotes the amplitude of the wave propagating forward (backward) in the positive (negative) x-direction and $k = \omega/v$ with constant velocity v.

We consider a system, which is composed of a material with a dielectric constant ϵ embedded between a and $b(a < b)$ and of vacuums (or airs) with a dielectric constant ϵ_0 situated in $x \leqslant a$ and $x \geqslant b$. $k/k_0 = \sqrt{\epsilon}/\sqrt{\epsilon_0}$ is a material constant. We assume ϵ to be real hereafter.

The continuity condition of the electromagnetic waves at $x = a$ gives the following relations among the amplitudes (A_r, A_l) in $x \leqslant a$ and (B_r, B_l) in $a \leqslant x \leqslant b$:

$$e^{ik_0a} A_r + e^{-ik_0a} A_l = e^{ika} B_r + e^{-ika} B_l, \tag{2}$$

$$k_0\, e^{ik_0a} A_r - k_0\, e^{-ik_0a} A_l = k\, e^{ika} B_r - k\, e^{-ika} B_l. \tag{3}$$

These are easily summarized as

$$\begin{pmatrix} B_r \\ B_l \end{pmatrix} = S^{-1}(k; a) \cdot S(k_0; a) \begin{pmatrix} A_r \\ A_l \end{pmatrix}, \tag{4}$$

using the matrix defined by

$$S(k; a) = \begin{pmatrix} e^{ika} & e^{-ika} \\ k\, e^{ika} & -k\, e^{-ika} \end{pmatrix} \tag{5}$$

and its inverse matrix $S^{-1}(k; a)$. In a similar way, for the amplitudes (C_r, C_l) in $x \geqslant b$ we have

$$\begin{pmatrix} C_r \\ C_l \end{pmatrix} = S^{-1}(k_0; b) \cdot S(k; b) \begin{pmatrix} B_r \\ B_l \end{pmatrix}. \tag{6}$$

Then one can give for the transfer matrix connecting the region of $x \leqslant a$ to that of $x \geqslant b$

$$T(b, a) = S^{-1}(k_0; b) \cdot S(k; b) \cdot S^{-1}(k; a) \cdot S(k_0; a), \tag{7}$$

whose elements, denoted by

$$T(b, a) = \begin{pmatrix} T_{rr} & T_{rl} \\ T_{lr} & T_{ll} \end{pmatrix}, \tag{8}$$

are

$$T_{rr} = e^{-ik_0(b-a)} \{k_{++}\, e^{ik(b-a)} + k_{--} e^{-ik(b-a)}\} = T_{ll}^*, \tag{9}$$

$$T_{rl} = e^{-ik_0(b+a)} \{k_{+-}\, e^{ik(b-a)} + k_{-+}\, e^{-ik(b-a)}\} = T_{lr}^*. \tag{10}$$

Such abbreviations as

$$\begin{aligned} k_{++} &= (1 + k/k_0)(1 + k_0/k)/4, \qquad & k_{+-} &= (1 + k/k_0)(1 - k_0/k)/4, \\ k_{-+} &= (1 - k/k_0)(1 + k_0/k)/4, \qquad & k_{--} &= (1 - k/k_0)(1 - k_0/k)/4 \end{aligned} \tag{11}$$

satisfy the following identities:

$$k_{++}k_{--} - k_{+-}k_{-+} = 0, \tag{12}$$

$$k_{+-} + k_{-+} = 0, \tag{13}$$

$$k_{++}^2 + k_{--}^2 - k_{+-}^2 - k_{-+}^2 = 1. \tag{14}$$

The elements of the transfer matrix depend not only on the width $(b-a)$ of the medium but also on its position $(b+a)$. We would like to emphasize that the arrangement of the factors in equations (9) and (10) is essential to avoid confusion in successive calculations.

We put the nth-stage pre-Cantor set in $[a, b]$, where $L = b - a$. From the left side the electromagnetic waves with the wave number $k_0 = \omega/v_0$ illuminate. For the amplitudes $\left(A_r^{(n)}, A_l^{(n)}\right)$ in $x \leqslant a$ and $\left(B_r^{(n)}, B_l^{(n)}\right)$ in $x \geqslant b$, we have

$$\begin{pmatrix} B_r^{(n)} \\ B_l^{(n)} \end{pmatrix} = T^{(n)}(b, a) \begin{pmatrix} A_r^{(n)} \\ A_l^{(n)} \end{pmatrix}. \tag{15}$$

Below, we will study the transmission ratio $t^{(n)}$ and the phase shift $\delta^{(n)}$, which are defined by

$$B_r^{(n)}\,\mathrm{e}^{\mathrm{i}k_0 L} = \sqrt{t^{(n)}}\,\mathrm{e}^{\mathrm{i}\delta^{(n)}}, \tag{16}$$

under the conditions of $A_r^{(n)} = 1$ and $B_l^{(n)} = 0$.

Due to the self-similarity of the triadic Cantor set, the transfer matrix $T^{(n)}$ for the nth stage is constructed from that for the previous $(n-1)$th stage with the identical shortest segments,

$$T^{(n)}(b, a) = T^{(n-1)}\left(b, \frac{2b+a}{3}\right) \cdot T^{(n-1)}\left(\frac{b+2a}{3}, a\right), \tag{17}$$

with $T^{(0)}(b, a) = T(b, a)$ in equation (8).

Let here $d_n = L/3^n$ and $\nu = k_0 L/9$. For the first-stage pre-Cantor set, $n = 1$, we have the respective elements of $T^{(1)}(b, a)$ through directly computing

$$T_{rr}^{(1)} = \mathrm{e}^{-\mathrm{i}k_0(b-a)}\left(\xi_1^2(d_1)\,\mathrm{e}^{3\mathrm{i}\nu} + |\eta_1(d_1)|^2\,\mathrm{e}^{-3\mathrm{i}\nu}\right) = T_{ll}^{(1)*}, \tag{18}$$

$$T_{rl}^{(1)} = \mathrm{e}^{-\mathrm{i}k_0(b+a)}(\xi_1(d_1)\,\mathrm{e}^{3\mathrm{i}\nu} + \xi_1^*(d_1)\,\mathrm{e}^{-3\mathrm{i}\nu})\eta_1(d_1) = T_{lr}^{(1)*}, \tag{19}$$

where

$$\xi_1(d) = k_{++}\,\mathrm{e}^{\mathrm{i}kd} + k_{--}\,\mathrm{e}^{-\mathrm{i}kd}, \tag{20}$$

$$\eta_1(d) = k_{+-}\,\mathrm{e}^{\mathrm{i}kd} + k_{-+}\,\mathrm{e}^{-\mathrm{i}kd}. \tag{21}$$

An identity, $|\xi_1(d)|^2 - |\eta_1(d)|^2 = 1$, derived by using equations (12) and (14), leads to $\left|T_{rr}^{(1)}\right|^2 - \left|T_{rl}^{(1)}\right|^2 = 1$.

Next we assume that the elements of $T^{(n)}(b, a)$ are given by

$$T_{rr}^{(n)} = \mathrm{e}^{-\mathrm{i}k_0(b-a)}\left(\xi_n^2(d_n)\,\mathrm{e}^{3\mathrm{i}\nu} + |\eta_n(d_n)|^2\,\mathrm{e}^{-3\mathrm{i}\nu}\right) = T_{ll}^{(n)*}, \tag{22}$$

$$T_{rl}^{(n)} = \mathrm{e}^{-\mathrm{i}k_0(b+a)}(\xi_n(d_n)\,\mathrm{e}^{3\mathrm{i}\nu} + \xi_n^*(d_n)\,\mathrm{e}^{-3\mathrm{i}\nu})\eta_n(d_n) = T_{lr}^{(n)*}. \tag{23}$$

Substituting above into equation (17) clarifies that the assumption is justified if

$$\xi_n(d_n) = \xi_{n-1}^2(d_n)\,\mathrm{e}^{\mathrm{i}\nu} + |\eta_{n-1}(d_n)|^2\,\mathrm{e}^{-\mathrm{i}\nu}, \tag{24}$$

$$\eta_n(d_n) = (\xi_{n-1}(d_n)\,\mathrm{e}^{\mathrm{i}\nu} + \xi_{n-1}^*(d_n)\,\mathrm{e}^{-\mathrm{i}\nu})\eta_{n-1}(d_n) \tag{25}$$

are satisfied with the initial terms $\xi_1(d_n)$ and $\eta_1(d_n)$. Dependence of $T^{(n)}(b, a)$ on the material constant $k/k_0 = \sqrt{\epsilon}/\sqrt{\epsilon_0}$ arises from $\xi_1(d_n)$ and $\eta_1(d_n)$. We note again

$$|\xi_n(d_n)|^2 - |\eta_n(d_n)|^2 = (|\xi_1(d_n)|^2 - |\eta_1(d_n)|^2)^{2^{n-1}} \\ = 1, \tag{26}$$

which yields

$$|T_{rr}^{(n)}|^2 - |T_{rl}^{(n)}|^2 = 1. \tag{27}$$

Although equations (24) and (25) are very complicated, the following transformations make them simple. We introduce

$$u_n = \{\xi_n(d_n)\,\mathrm{e}^{\mathrm{i}\nu} + \xi_n^*(d_n)\,\mathrm{e}^{-\mathrm{i}\nu}\}/2, \tag{28}$$

$$v_n = -\mathrm{i}\{\xi_n(d_n)\,\mathrm{e}^{\mathrm{i}\nu} - \xi_n^*(d_n)\,\mathrm{e}^{-\mathrm{i}\nu}\}/2 \tag{29}$$

and

$$p_n = \{\eta_n(d_n) + \eta_n^*(d_n)\}/2, \tag{30}$$

$$q_n = -\mathrm{i}\{\eta_n(d_n) - \eta_n^*(d_n)\}/2. \tag{31}$$

Their initial values are

$$u_1 = K_n \cos(\nu + \theta_n), \tag{32}$$

$$v_1 = K_n \sin(\nu + \theta_n), \tag{33}$$

$$p_1 = (k_{+-} + k_{-+}) \sin(kd_n) = 0, \tag{34}$$

$$q_1 = \kappa_- \sin kd_n, \tag{35}$$

where

$$K_n = \sqrt{\cos^2 kd_n + \kappa_+^2 \sin^2 kd_n} \geqslant 1, \tag{36}$$

$$\tan\theta_n = \frac{\kappa_+ \sin kd_n}{\cos kd_n} \tag{37}$$

and

$$\kappa_+ = \left(\frac{k}{k_0} + \frac{k_0}{k}\right)\bigg/2 \geqslant 1, \tag{38}$$

$$\kappa_- = \left(\frac{k}{k_0} - \frac{k_0}{k}\right)\bigg/2 > 0. \tag{39}$$

For later use we divide the ω region into Ω_1, Ω_2 and Ω_3, where

$$\begin{aligned} &\omega \in \Omega_1 \quad \text{if} \quad |K_n \cos(\nu + \theta_n)| \leqslant 1, \\ &\omega \in \Omega_2 \quad \text{if} \quad K_n \cos(\nu + \theta_n) > 1, \\ &\omega \in \Omega_3 \quad \text{if} \quad K_n \cos(\nu + \theta_n) < -1, \end{aligned} \tag{40}$$

Substituting equations (28)–(31) into equations (24) and (25), we can straightforwardly derive following recurrence equations, for $n \geqslant 2$:

$$u_n = 2u_{n-1}^2 - |\xi_{n-1}(d_n)|^2 + |\eta_{n-1}(d_n)|^2 = 2u_{n-1}^2 - 1, \tag{41}$$

$$v_n = 2u_{n-1}v_{n-1}, \tag{42}$$

$$p_n = 2u_{n-1}p_{n-1}, \tag{43}$$

$$q_n = 2u_{n-1}q_{n-1}. \tag{44}$$

The solution of equation (41) is immediately obtained by transforming $u_n = \cos\phi_n$ or $\cosh\phi_n$ corresponding to $|u_n| \leqslant 1$ or $u_n > 1$, respectively, for $n \geqslant 2$,

$$u_n = \begin{cases} \cos(2^{n-1}\phi_1) & \text{for } \omega \in \Omega_1, \\ \cosh(2^{n-1}\phi_1) & \text{for } \omega \in (\Omega_2 \cup \Omega_3), \end{cases} \tag{45}$$

where ϕ_1 is determined by

$$u_1 = K_n \cos(\nu + \theta_n) = \begin{cases} \cos\phi_1 & \text{for } \omega \in \Omega_1, \\ \cosh\phi_1 & \text{for } \omega \in \Omega_2. \end{cases} \tag{46}$$

If $u_1 > 1$ and even if $u_1 < -1, u_n$ is kept to be larger than 1 for $n \geqslant 2$. When $|u_1| \leqslant 1, |u_n| \leqslant 1 (n \geqslant 2)$.

Next, equation (42) easily gives

$$v_n = 2^2 u_{n-1}u_{n-2}v_{n-2} = \cdots = 2^{n-1}v_1 \prod_{j=1}^{n-1} u_j. \tag{47}$$

Similarly we have

$$p_n = 2^{n-1}p_1 \prod_{j=1}^{n-1} u_j = 0, \tag{48}$$

$$q_n = 2^{n-1}q_1 \prod_{j=1}^{n-1} u_j. \tag{49}$$

Since the rigorous expression of the transfer matrix $T^{(n)}(b, a)$ can be obtained successfully, we can extract any information of the electromagnetic waves propagating through the nth-stage pre-Cantor set. In this letter, our interests are limited to the transmission ratio $t^{(n)}$ and the phase shift $\delta^{(n)}$. Using equation (15) under the conditions of $A_r^{(n)} = 1$ and $B_l^{(n)} = 0$, we have

$$B_r^{(n)} = \left(\left|T_{rr}^{(n)}\right|^2 - \left|T_{rl}^{(n)}\right|^2\right) / T_{ll}^{(n)} = 1 / T_{ll}^{(n)}. \tag{50}$$

Then from the definition (16), the transmission ratio $t^{(n)}$ and the phase shift $\delta^{(n)}$ are given as

$$t^{(n)} = \left|B_r^{(n)}\right|^2 = \frac{1}{1 + 4q_n^2(u_n \cos 2\nu - v_n \sin 2\nu)^2}, \tag{51}$$

and

$$\delta^{(n)} = \nu + \psi^{(n)}, \tag{52}$$

respectively, where

$$\tan\psi^{(n)} = \frac{2u_n v_n - q_n^2 \sin 4\nu}{u_n^2 - v_n^2 + q_n^2 \cos 4\nu}. \tag{53}$$

In this case, the reflection ratio $r^{(n)} = |A_l^{(n)}|^2$ is given by $r^{(n)} = 1 - t^{(n)}$ because equation (27) holds.

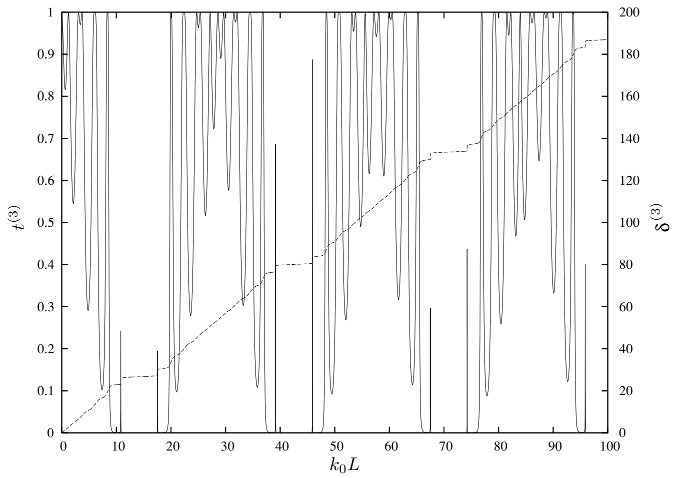

Figure 1. The transmission spectrum $t^{(3)}$ (left axis, solid curve) and the phase shift $\delta^{(3)}$ (right axis, broken curve) for the third-stage pre-Cantor set as functions of k_0L.

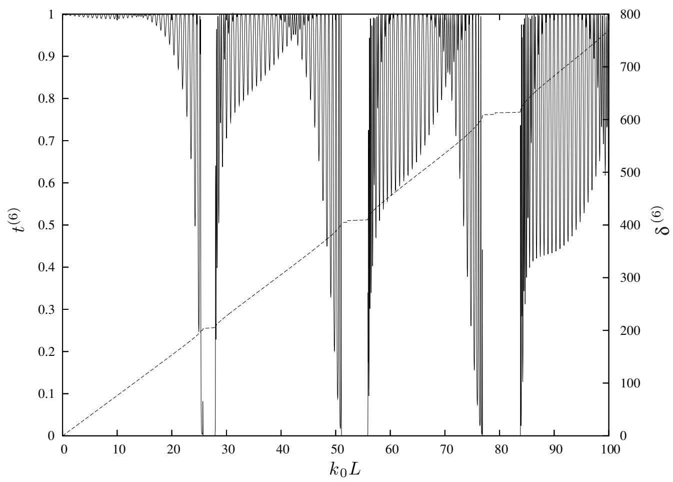

Figure 2. The transmission spectrum $t^{(6)}$ (left axis, solid curve) and the phase shift $\delta^{(6)}$ (right axis, broken curve) for the sixth-stage pre-Cantor set as functions of k_0L.

One can easily compute numerically, without any limitations, the transmission spectrum and the phase shift for the arbitrary-stage pre-Cantor set by substituting equations (45), (47) and (49) into equations (51) and (52). Simultaneously, the tests whether the solution in equation (45) is unique or not have been performed by solving numerically equation (41). We take $\epsilon/\epsilon_0 = 8.8$ for an epoxy resin mixed with metal oxides [8].

For relatively low stages, the spectra are rather complicated as reported in previous papers [11–13] and as shown in figure 1, where $t^{(3)}$ and $\delta^{(3)}$ are displayed as functions of $k_0L = \omega L/v_0$ with the light velocity v_0. The number of computed points for $0 \leqslant k_0L \leqslant 100$ is 2^{12}.

One finds some masked regions where $t^{(3)}$ is not absolute but almost zero. The masked regions correspond nearly to Ω_2 or Ω_3 defined in equation (40), because the condition $|u_1| > 1$

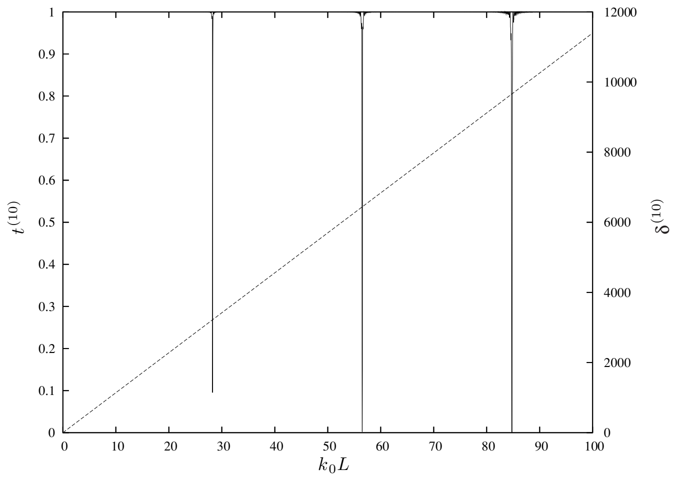

Figure 3. The transmission spectrum $t^{(10)}$ (left axis, solid curve) and the phase shift $\delta^{(10)}$ (right axis, broken curve) for the tenth-stage pre-Cantor set as functions of k_0L.

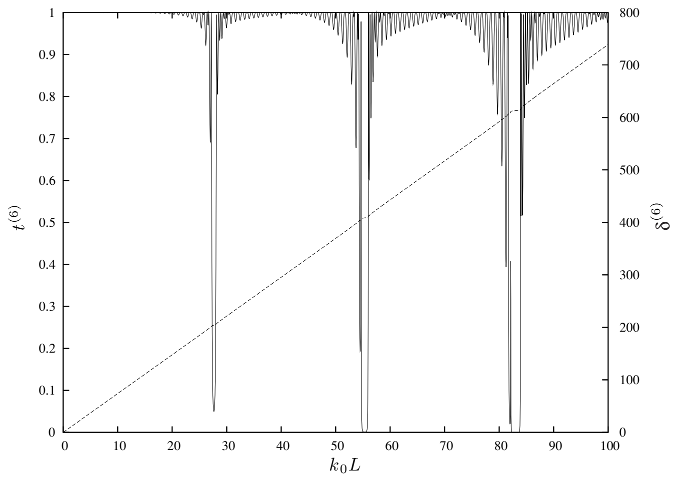

Figure 4. The transmission spectrum $t^{(6)}$ (left axis, solid curve) and the phase shift $\delta^{(6)}$ (right axis, broken curve) for the sixth-stage pre-Cantor set as functions of k_0L in the case of $\epsilon/\epsilon_0 = 2.8$.

leads to the almost total reflection. The spectra do not show the self-similarity particular to fractals. The reason is because the electromagnetic waves propagate not only through the media with the self-similar symmetry, but also through the complementary spaces (the airs) without the symmetry.

As the stage is increased the masked regions become narrower step-by-step as shown in figures 2–3. Eventually only lines like hanged ropes can be seen. Their positions are given by $\cos\nu = \pm 1$ or $k_0L = 9\pi\ell$, (ℓ = any positive integers), which are the boundaries between Ω_1 and Ω_2 or between Ω_1 and Ω_3, because K_n and θ_n tend to unity and 0, respectively, as n is increased. However, the hanged ropes rise slowly from the left, that is, the values of $t^{(n)}$ at $k_0L = 9\pi\ell$ increase gradually in the order from smaller ℓ as shown initially in figure 3. Although the phase shift $\delta^{(10)}$ seems to increase smoothly with a constant slope, its derivative changes drastically at $k_0L = 9\pi\ell$, as examined numerically. Finally in the limit of $n \to \infty$, $t^{(n)}$ is expected to take unity everywhere. In this limit the transfer matrix $T^{(n)}(b, a)$ tends to the unit matrix, because the Lebesgue measure of the true Cantor set vanishes [17].

At last, for comparison, we display the computation result in figure 4, obtained by changing the dielectric constant into $\epsilon/\epsilon_0 = 2.8$ corresponding to an epoxy resin [7]. The narrowing and rising of the dips occur at relatively lower stages.

In conclusion, we have succeeded in obtaining the rigorous expression of the transfer matrix for an arbitrary-stage pre-Cantor set. By using it, the transmission spectrum and the phase shift have been calculated exactly. Numerical computations have revealed the hidden periodicity in the self-similar symmetry, which leads to the first theoretical description of the sharp attenuation in the transmission spectrum of electromagnetic waves propagating through a fractal medium, as observed in the experiment [7, 8].

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